

On a sum of centered random variables with nonreducing variances.

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Abstract. Let $x = (x_1, \dots, x_n)$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n$. If $\|x\|_2 = 1$ and the coordinates ε_i of ε are independently distributed random variables with $Pr(\varepsilon_i = 1) = Pr(\varepsilon_i = -1) = \frac{1}{2}$, then

$$Pr(|\varepsilon^T x| \leq 1) \geq 0.36$$

We give a new proof of this inequality which is weaker than the best known one $Pr(|\varepsilon^T x| \leq 1) \geq \frac{3}{8}$, proved by R. Holzman and D.J. Kleitman.

Introduction. Let us consider $x = (x_1, \dots, x_n)$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n$ such that $\|x\|_2 = 1$ and the coordinates ε_i of ε are independently distributed random variables with $Pr(\varepsilon_i = 1) = Pr(\varepsilon_i = -1) = \frac{1}{2}$.

The problem is to give low bounds for $Pr(|\varepsilon^T x| \leq 1)$ for arbitrary x . R. Holzman and D.J. Kleitman in [1] proved that $Pr(|\varepsilon^T x| \leq 1) \geq \frac{3}{8}$ and conjectured that $Pr(|\varepsilon^T x| \leq 1) \geq \frac{1}{2}$. For comparison the strong inequality $Pr(|\varepsilon^T x| < 1) \geq \frac{3}{8}$ is sharp. There are various bounds in [2], see it also for references.

Developing robust optimization technology A. Ben-Tal, A. Nemirovski and C. Roos have had to prove that $Pr(|\varepsilon^T x| \leq 1)$ is restricted from zero to proceed an effective bound of the "level of conservativeness", see [3] for details. It was proved in [3] that $Pr(|\varepsilon^T x| \leq 1) \geq \frac{1}{3}$ (result of R. Holzman and D.J. Kleitman was unknown yet).

In the present communication we shall show how to upgrade arguments of [3] to prove that $Pr(|\varepsilon^T x| \leq 1) \geq 0.36$. We hope that this bound may be improved in a similar way up to more than $\frac{3}{8}$.

Theorem 1. Let $x = (x_1, \dots, x_n)$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{R}^n$. If $\|x\|_2 = 1$ and the coordinates ε_i of ε are independently distributed random variables with $Pr(\varepsilon_i = 1) = Pr(\varepsilon_i = -1) = \frac{1}{2}$, then

$$Pr(|\varepsilon^T x| \leq 1) \geq 0.36$$

Proof. Without loss of generality we may assume that

$$x_1 \geq x_2 \geq \dots \geq x_n \geq 0.$$

Let us consider two cases.

Case 1, $x_1 + x_2 > 1$. Let us denote

$$r = r(\varepsilon) = \varepsilon_3 x_3 + \dots + \varepsilon_n x_n$$

The random quantity r has symmetric distribution and hence

$$\begin{aligned} Pr(|\varepsilon^T x| \leq 1) &= \frac{1}{2} Pr(-1 \leq r + x_1 + x_2 \leq 1) + \frac{1}{2} Pr(-1 \leq r + x_1 - x_2 \leq 1) = \\ &= \frac{1}{2} (Pr(-1 - x_1 - x_2 \leq r \leq 1 - x_1 - x_2) + Pr(-1 - x_1 + x_2 \leq r \leq 1 - x_1 + x_2)) = \\ &= \frac{1}{2} (Pr(0 \leq r \leq 1 + x_1 + x_2) + Pr(0 < r \leq 1 + x_1 - x_2)) + \\ &\quad + \frac{1}{2} Pr(x_1 + x_2 - 1 \leq r \leq 1 - x_1 + x_2) \geq \\ &\geq \frac{1}{4} (Pr(|r| \leq 1 + x_1 + x_2) + Pr(|r| \leq 1 + x_1 - x_2)). \end{aligned} \tag{1}$$

By the Tschhebyshev inequality

$$Pr(|r| \leq 1 + x_1 - x_2) > 1 - \frac{Mr^2}{(1 + x_1 - x_2)^2} \geq \frac{1}{2}, \quad (2)$$

since $x_1 \geq x_2$ and

$$Mr^2 = 1 - x_1^2 - x_2^2 \geq \frac{1}{2}$$

for $x_1 + x_2 \geq 1$.

By the 4-th degree Tschhebyshev inequality we have

$$Pr(|r| \leq 1 + x_1 + x_2) > 1 - \frac{Mr^4}{(1 + x_1 + x_2)^4} = 1 - \frac{1}{(1 + x_1 + x_2)^4} \left(\sum_{i=3}^n x_i^4 + 6 \sum_{3 \leq i < j \leq n} x_i^2 x_j^2 \right).$$

Notice that $1 + x_1 + x_2 \geq 2$ and

$$\sum_{i=3}^n x_i^4 + 6 \sum_{3 \leq i < j \leq n} x_i^2 x_j^2 < 3(x_3^2 + \dots + x_n^2)^2 < \frac{3}{4}.$$

Hence

$$Pr(|r| \leq 1 + x_1 + x_2) > 1 - \frac{3}{64}. \quad (3)$$

So in the case $x_1 + x_2 > 1$ inequalities (1), (2) and (3) yield

$$Pr(|\varepsilon^T x| \leq 1) \geq \frac{93}{256} > 0.36$$

Case 2. $x_1 + x_2 \leq 1$. Let us denote

$$s_k = s_k(\varepsilon) = \varepsilon_1 x_1 + \dots + \varepsilon_k x_k$$

for $1 \leq k \leq n$. Let us define the following events

$$A_k = \{\varepsilon : |s_j(\varepsilon)| \leq 1 - x_{j+1}, j = 1, \dots, k-1, \text{ and } |s_k(\varepsilon)| > 1 - x_{k+1}\}$$

for $2 \leq k \leq n-1$ and

$$A_n = \{\varepsilon : |s_j(\varepsilon)| \leq 1 - x_{j+1}, j = 1, \dots, n-1\}.$$

In the current case the events A_2, \dots, A_n form a partition of probability space. In order to prove

$$Pr(|s_n| \leq 1) \geq 0.36$$

we shall get

$$Pr(|s_n| \leq 1 \mid A_k) \geq 0.36$$

provided $Pr(A_k) > 0$. Surely $Pr(|s_n| \leq 1 \mid A_n) = 1$. If $\varepsilon \in A_k$ for $2 \leq k \leq n-1$, then $|s_k(\varepsilon)| \leq 1$ and

$$Pr(|s_n| \leq 1 \mid A_k) \geq Pr(0 \leq s_n - s_k \leq 2 - x_{k+1}) \geq \frac{1}{2} Pr(|s_n - s_k| \leq 2 - x_{k+1}) \geq \frac{1}{2} \left(1 - \frac{M(s_n - s_k)^2}{(2 - x_{k+1})^2} \right) \quad (4)$$

by Tschhebyshev inequality. Note that

$$M(s_n - s_k)^2 = 1 - x_1^2 - \dots - x_k^2 \leq 1 - kx_{k+1}^2. \quad (5)$$

Let us define a function

$$g_k(x) = \frac{1}{2} \left(1 - \frac{1 - kx^2}{(2 - x)^2} \right).$$

It follows from (4) and (5) that

$$Pr(|s_n| \leq 1 \mid A_k) \geq g_k(x_{k+1}). \quad (6)$$

Since $\varepsilon \in A_k$, so

$$1 - x_{k+1} \leq x_1 + x_2 + \dots + x_k.$$

By the Cauchy inequality

$$\frac{x_1 + x_2 + \dots + x_k}{k} \leq \sqrt{\frac{x_1^2 + \dots + x_k^2}{k}} = \sqrt{\frac{1 - M(s_n - s_k)^2}{k}}.$$

Hence

$$M(s_n - s_k)^2 \leq 1 - \frac{(1 - x_{k+1})^2}{k}. \quad (7)$$

Let us define a function

$$h_k(x) = \frac{1}{2} \left(1 - \frac{1 - \frac{(1-x)^2}{k}}{(2-x)^2} \right)$$

It follows from (4) and (7) that

$$Pr(|s_n| \leq 1 \mid A_k) \geq h_k(x_{k+1}). \quad (8)$$

Now we are going to analyse functions $g_k(x)$ and $h_k(x)$ for $x \in [0, 1]$ and $k \geq 2$. It is easy to check that $g_k(x)$ increases for $x \in [\frac{1}{2k}, 1]$ and that $h_k(x)$ decreases for $x \in [0, 1]$. Moreover, $g_k(x) = h_k(x) \Leftrightarrow x = \frac{1}{k+1}$ and

$$\min_{x \in [0, 1]} \max \{g_k(x), h_k(x)\} = g_k\left(\frac{1}{k+1}\right).$$

This allows us to deduce from (6) and (8) the following

$$Pr(|s_n| \leq 1 \mid A_k) \geq \max \{g_k(x_{k+1}), h_k(x_{k+1})\} \geq g_k\left(\frac{1}{k+1}\right) \geq g_2\left(\frac{1}{3}\right) = 0.36.$$

Since the events A_2, \dots, A_n form a partition of probability space, we finally have

$$Pr(|s_n| \leq 1) \geq 0.36.$$

Remark 1. One may use 4—th degree Tschhebyshev inequality in the case 2 part of the proof instead of (4). It may lead to a better bound and I suppose, that $p \geq 0.4$ is possible to prove in a such way.

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References

- [1] R. Holzman and D. J. Kleitman, *On the product of sign vectors and unit vectors*, Combinatorica **12** (1992), no. 3, 303–316.
- [2] M. Veraar *A note on optimal probability lower bounds for centered random variables*, arXiv:0803.0727v2
- [3] A. Ben-Tal, A. Nemirovski, C. Roos, *Robust solutions to uncertain quadratic and conic-quadratic problems*, SIAM J. on Optim. 2002, **13**, 535-560